$$v^2 = v_x^2 + v_y^2 = v_r^2 + v_\theta^2. \quad (7)$$

On substituting Eqs. (1) and (2) into Eq. (7) we get

$$v^2 = v_h^2 [1 + e^2 + 2e \cos\theta]. \quad (8)$$

Using Eq. (3) for v_y also in Eq. (7) and equating the two results for v^2 yields the hodograph in the form

$$v_x^2 + (v_y - v_p)^2 = v_h^2. \quad (9a)$$

Since all symbols are components of velocity vectors we take Eq. (9a) to be the basic form of the velocity hodograph. It shows that the radius of the velocity circle is v_h and that its center is displaced by v_p on the y axis. This form of the hodograph is independent of A and it underscores the natural role that the two invariant velocities play in the Kepler problem.[8]

Now, from Ref. 5, $v_h = k/L = GMm/L$ in which L is the magnitude of the orbital angular momentum of Earth, $k = GMm$ is the force constant where G is the gravitational constant and M,m are, respectively, the mass of the Sun and the Earth. Furthermore, from Ref. 5, $v_p = A/mL$, where A is the magnitude of the LRL vector.[9] To conform to the notation of PDN we call $L = J$ and get

$$v_x^2 + (v_y - A/mJ)^2 = (GmM/J)^2. \quad (9b)$$

From this equation,[10] which we call the dynamical form of the hodograph, Noerdlinger stated that the LRL vector **A** "is thus identified as a well known quantity in practical astronomy." While this is correct, it appears that the invariant velocities are revealed more directly and simply.

In terms of measurable quantities e, K, and c, since $v_p = eKc$ and $v_h = Kc$, the hodograph is

$$v_x^2 + (v_y - eKc)^2 = (Kc)^2, \quad (9c)$$

which, since it depends on K, we call the stellar aberration form.

In sum, the aberration constant as defined by Eq. (6) is simply the ratio of the constant rotating speed v_h, the radius of the hodograph for Earth, to the speed of light; hence, $v_h = Kc$. The displacement of the center of the hodograph is the invariant velocity $v_p = ev_h = A/mL = eKc$. Therefore, measurements of stellar aberration, the velocity of light, and the orbital eccentricity suffice in principle for a determination of the two invariant velocities. The displacement of the center of the aberration circle noted by Noerdlinger is seen to be v_p in units of the velocity of light.

In this method additional measurements of the angular momentum L and Earth mass m would in principle be required to define the magnitude of the LRL vector **A**. The simple form shown by Eq. (9a) suggests that among the various velocities associated with the Kepler problem the two invariant velocities are special, even primal, and that $\mathbf{v}_p$ rather than $\mathbf{A}/mL$ is fundamental on the basis of simplicity. This form should be of interest in a mechanics course. The form, Eq. (9c), for the hodograph suggests a blend of mechanical and optical parameters that may be of interest to astronomers.

Finally, the attention focused by Noerdlinger on the LRL vector while ignoring $\mathbf{p}_p$ is not an uncommon practice of authors in this Journal and in recent mechanics textbooks. It does not detract from the basic insights in Ref. 1 that link mechanical and optical planetary phenomena.

[a] Professor Emeritus.

[1] Peter D. Noerdlinger, "Stellar aberration and the hodograph for the Kepler problem," Am. J. Phys. **45**, 1229–1230 (1977).

[2] Herbert Goldstein, "More on the prehistory of the Laplace or Runge–Lenz vector," Am. J. Phys. **44**, 1123–1124 (1976). The momentum hodograph used is the same as the velocity hodograph multiplied by a scale factor of the mass. For a discussion of the LRL vector, see Herbert Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980), 2nd ed., pp. 102–105.

[3] W. M. Smart, *Textbook On Spherical Astronomy* (Cambridge U.P., Cambridge, 1986), 6th ed., p. 181.

[4] Reference 3.

[5] Ferdinand J. Shore, "The Kepler problem recast: Use of a transverse velocity transformation and the invariant velocities," Am. J. Phys. **55**, 139–146 (1987).

[6] Reference 5. See Eqs. (32) and (34) which yield this form for T since $R = a(1 - e^2)$.

[7] Reference 3, pp. 185 and 191; Ref. 1, Eq. (1).

[8] Reference 5. See. Eq. (13) in which the vector velocity is expressed as $\mathbf{v} = v_h \hat{\theta} + v_p \hat{j}$, where $\hat{\theta}$ and $\hat{j}$ are, respectively, unit vectors perpendicular to **r** and perpendicular to the axis through perihelion. See also J. M. A. Danby, *Fundamentals of Celestial Mechanics* (Macmillan, New York, 1962), p. 131; Ref. 3, pp. 110, 185, and 190 in which the invariant velocity contributions to the aberration are discussed without reference to the hodograph. The invariant velocities from a geometrical point of view are discussed by Harold Abelson, Andrea diSessa, and Lee Rudolph, Am. J. Phys. **43**, 579–589 (1975).

[9] Reference 5. See Eq. (16); Ref. 2, pp. 102–105.

[10] See Ref. 1, Eq. (4), which was derived by PDN from papers by Herbert Goldstein, Ref. 2, and Lloyd Motz, "The conservation principles and Kepler's laws of planetary motion," Am. J. Phys. **43**, 575 (1975).

Wave cutoff on a suspended slinky

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A long slinky, suspended from above by strings, makes a beautiful demonstration of wave propagation (see Fig. 1). If the slinky's supporting strings are short, then wave cutoff results, meaning that waves do not propagate below a critical frequency. Transverse waves are driven by a heavy variable-length pendulum attached to one end of the

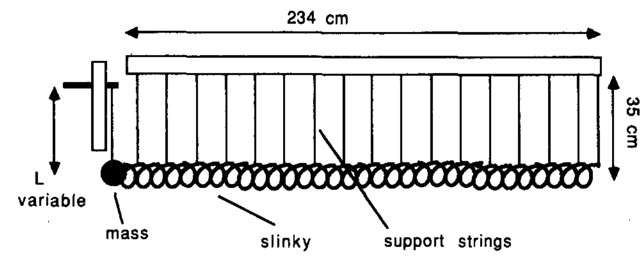

Fig. 1. Sketch of the hanging slinky and pendulum driver. The pendulum motion is normal to the plane of the figure. The length of the support strings L_s remains fixed, while the length of the pendulum L can be varied.

slinky, so that the driving frequency is set by the length L of the pendulum. For most frequencies, the pendulum has enough stored energy to drive the slinky for many periods of oscillation. Waves are excited by simply setting the pendulum in motion. At higher frequencies, it is necessary to "pump" the pendulum at each cycle with a gentle hand motion. It is also possible to produce amplitude modulation and wave packets by gradually changing the pendulum's amplitude.

The horizontal displacement of the slinky $y(x,t)$ obeys

$$\frac{\partial^2 y}{\partial t^2} = v_0^2 \frac{\partial^2 y}{\partial x^2} - \omega_0^2 y, \qquad (1)$$

where v_0 is the asymptotic wave velocity at high frequency. The term containing ω_0^2 is due to the supporting strings. A derivation of Eq. (1) starts with the standard deviation[1] of the wave equation for a taut string: A small element of string of length dx has mass $\rho\, dx$ and experiences a restoring force of $Td(dy/dx)$, where T is the tension and y is the displacement. If the slinky is supported from above by strings of length L_s, then there is an additional restoring force due to gravity. The transverse component of this gravitational force on the element is $(\rho\, dx)gy/L_s$, provided $y \ll L_s$. Equating the sum of the two forces (gravity plus tension) to the acceleration $ma = (\rho\, dx)(d^2y/dt^2)$ yields Eq. (1), where $\omega_0^2 = g/L_s$ and $v_0^2 = T/\rho$.

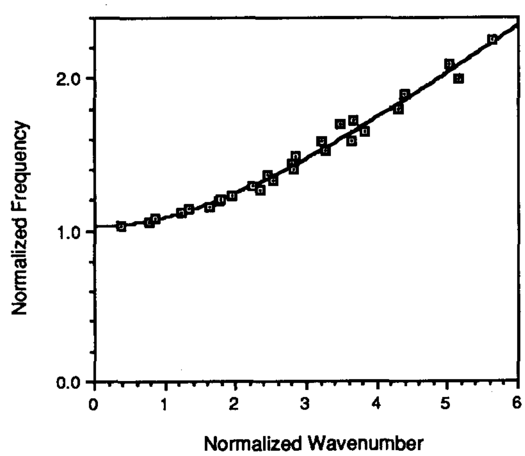

Fig. 2. Dispersion relation $\omega(k)$ of the slinky. The normalized frequency is $\omega/\omega_0 = (L_s/L)^{1/2}$. The normalized wavenumber $234/\lambda$ is the number of wavelengths contained in the slinky. The solid line is a hyperbola, which was fit to Eq. (2), with $\omega_0 = 5.47\ \text{s}^{-1}$ and $v_0 = 69.4$ cm/s giving the best fit.

The driven slinky produces a standing wave $y = \cos(kx)\sin(\omega t)$, where

$$\omega^2 = \omega_0^2 + k^2 v_0^2 \qquad (2)$$

is the same dispersion relation obeyed by electromagnetic radiation in waveguides and optical fibers, and by Langmuir waves in a plasma.[2-4]

In order to measure the dispersion relation, we control the driving frequency $\omega = (g/L)^{1/2}$ by varying the pendulum length L. Values of wavelength $\lambda = 2\pi/k$ were obtained by photographing the moving slinky and measuring distances between nodes and antinodes. The opposite end of the slinky was usually allowed to hang free, allowing measurements of wavelengths longer than the slinky. Although string at the free end of the slinky did not hang vertically, the resulting shrinkage of the slinky was acceptable and can even be used to make a crude estimate of the slinky's tension.[5]

Experimental values of $\omega(k)$ are shown in Fig. 2 along with a hyperbola that was fit to the data. The least-squares fit of the data to Eq. (2) gave $\omega_0 = 5.47\ \text{s}^{-1}$ and $v_0 = 69.4$ cm/s, with an uncertainty of about 6%. The measured value of ω_0 differs by 3% from the theoretical value of $\omega_0 = (g/L_s)^{1/2}$. The value of v_0 obtained from the dispersion relation allowed us to calculate the tension in the slinky. Using $T = \rho v_0^2$, and a measured value of $\rho = 2.05$ g/cm, we obtained $T = 9.9 \times 10^3$ dyn ($\pm 12\%$).

We also made a direct measurement of the slinky's tension by holding a slinky vertically, allowing several loops to hang below.[6] We then picked a point on the slinky where the spacing between loops corresponds to the experimental conditions (1.45 cm), and determined the mass m' of the slinky below that point. Using $T = m'g$, we obtained 9.8×10^3 dyn ($\pm 4\%$), which is almost in exact agreement with the measurement based on the dispersion relation.

A few hints about constructing the device: The 234-cm-long slinky consisted of four plastic slinkies[7] taped together. The strings supported the slinky 35 cm below the horizontal rod, were spaced 7.25 cm apart, and were attached at every fifth loop of the slinky. The loop spacing was 1.45 cm. The driving pendulum consisted of a heavy twine attached to a sphere of mass 1743 g and radius 3.9 cm. The driving pendulum was adequate: A larger mass would be better, but a larger radius would preclude excitation at high frequencies. (For example, we could not make L less than 4 cm.) The use of a polaroid camera is not necessary if one is willing to measure the wavelength directly by hand. It is recommended that the total length of the slinky be much larger: A slinky three times as long would extend the full length of a typical classroom. For careful measurements, one might consider decreasing the number of slinky loops per support string from the ratio 5:1 used on our device.

The pendulum driver is strongly recommended. An earlier attempt with a crude motor-driven assembly failed due to the excitation of higher harmonics. Also, students can immediately grasp the physics of the device because the slinky exhibits cutoff when the length of the driving pendulum equals the length of the supporting strings.

[1]A. P. French, *Vibrations and Waves* (Norton, New York, 1971), p. 162.
[2]P. Lorrain and D. Corson, *Electromagnetic Fields and Waves* (Freeman, San Francisco, 1970), pp. 489 and 571.

[3] Also Alan Portis, *Electromagnetic Fields: Sources and Media* (Wiley, New York, 1978), pp. 171, 515–526.

[4] F. F. Chen, *Introduction to Plasma Physics* (Plenum, New York, 1974).

[5] The free end of the slinky was shrunk by a length $S = 8$ cm less than the length determined by the supporting strings, with the shrinkage confined to the end of the slinky. The tension is approximately $T = \rho g S^2/L_s = 4\times 10^3$ dyn. This formula is valid when the shrinkage S is small, and can be derived by converting difference equations for the shrinkage of the hanging slinky into a differential equation. Another way to estimate the tension is to measure the exponential decay of the slinky's transverse displacement $y(x) = y_0 \exp(-x\omega_0/v_0)$ when the pendulum is held motionless at fixed amplitude. This method yielded $T = 1.5\times 10^4$ dyn. Both of these estimates of T were not very accurate because the slinky shrinkage and the string spacing were not sufficiently small for the equations to be valid.

[6] S. Y. Mak, Am. J. Phys. **55**, 994–947 (1987).

[7] Each plastic slinky had 38 loops, weighed 113 g, had a diameter of 8 cm, and was made by James Industries, Inc., Hollidaysburg, PA, 16648.

A qualitative demonstration of the conservation of angular momentum in a system of two noncoaxial rotating disks

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I. INTRODUCTION

The principle of conservation of angular momentum is the rotational counterpart of the principle of conservation of linear momentum. According to the principle, the angular momentum of a rotating system remains constant provided no external torques act on it. Unlike the conservation of linear momentum that can be demonstrated readily using a system of interacting bodies, the conservation of angular momentum is usually demonstrated, as described in the literature and in most texts, using a system with only one body having a variable moment of inertia that can be made to spin about a fixed axis of rotation.[1] The change in angular speed ω is observed as we vary the moment of inertia I of the body, the two physical quantities obey a simple relationship of indirect proportion, namely, angular momentum $= L = I\omega =$ constant. Since accurate measurements of both angular speed and moment of inertia require sophisticated equipment and are very time consuming, most demonstrations of this relation are qualitative in nature. Satisfactory results can often be obtained if damping is small and the duration of the experiment is reasonably short.

A demonstration of this principle involving a system of two interacting bodies with a common axis of rotation was also described in Meiner's book.[1] A ball-bearing nut is allowed to fall down from a freely suspending vertical bolt. The two bodies turn in opposite directions during the period of falling. This experiment can be considered as the angular version of the standard mechanics problem—*A ball rolls down a tilted wedge that is placed on a frictionless horizontal plane...*—in which the wedge moves backward as the ball rolls forward down the incline. Unfortunately, the ball-bearing nut-and-bolt system used in this experiment is not only expensive but also too uncommon to be available in ordinary hardware stores.

In this note, we describe two experiments using a single device that demonstrate also the conservation of angular momentum of a system of two interaction bodies. The new design, however, serves some purposes different from that of the ball-bearing nut-and-bolt experiment. It has the following interesting special features: (1) The bodies do not rotate about a common axis; (2) the nature of the coupling between the two bodies can be changed depending on the initial setting; (3) the finished product is large enough to be clearly visible at the back of the demonstration hall; and (4) the materials for construction are easily available.

II. STRUCTURE

The system consists of two standard-sized 30-cm-diam phonograph turntables (Fig. 1). The first (A) is a high-quality turntable originally designed to be pivoted by an air cushion[2] at the center O. It has been disengaged from its motor to ensure that the solid friction acting at the shaft is reduced to a minimum. The second (B) is an arbitrary turntable mounted on the shaft O′ of a 0- to 12-V dc motor which by itself is mounted noncoaxially on the top of A. A 6-V battery box that powers the dc motor is also mounted on the top of A, on the side opposite to the motor, acting also as a counterweight. The position of the battery box must be carefully adjusted to restore the balance of the loaded turntable about its pivot O to prevent side friction effects at the shaft. The maintenance of a "friction-free" state of A is crucial for a successful demonstration.

III. THEORY AND OPERATION

The demonstration can be divided into two parts. In the first experiment, both turntables are initially at rest. When we switch on the motor, the two turntables will rotate in opposite directions with B about the moving axis O′ and

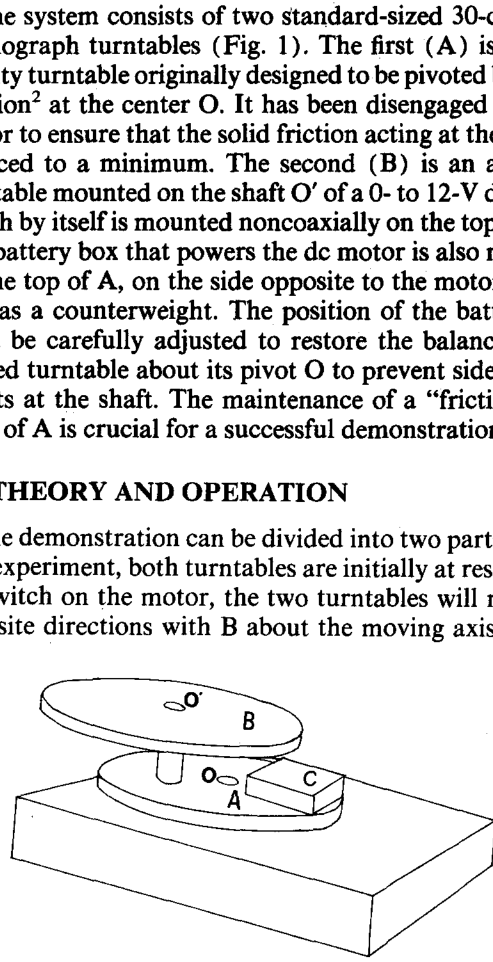

Fig. 1. The construction of the conservation of angular momentum apparatus. A—The frictionless turntable; B—The motor-driven turntable; and C—The battery box (acting also as the counterweight).