Deducing the width of a Lorentzian resonance curve from experimental data

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A quick but reliable method for determining the width of a resonance curve from experimental data is described here. We consider data that fit either a true Lorentzian curve, or the "near" Lorentzian obtained from a classical driven harmonic oscillator.[1,2]

Suppose first that we wish to deduce γ from experimental data that is described by a true Lorentzian curve:

$$L(\omega)=\frac{1}{(\omega-\omega_0)^2+\gamma^2}, \tag{1}$$

where ω is the driving frequency and ω_0 is the resonant frequency. The "damping term" γ is both a measure of the width of the curve, and the degree to which the system is damped. It is obvious from (1) that 2γ equals the FWHM (full-width-at-half-maximum), defined as the difference between the two values of ω for which $L(\omega)$ falls to half its maximum value. The problem is that a simple determination of FWHM from experimental data does not use all the available information.

A superior way to find γ is to measure the width of the peak at a number of places. Let $\Delta\omega$ be the difference between two values of ω where $L(\omega)$ falls below its peak value by a factor $R<1$. It follows from Eq. (1) that

$$\gamma=\sqrt{\frac{R}{1-R}}\,\frac{\Delta\omega}{2}, \tag{2}$$

where $R=L(\omega)/L(\omega_0)$ with ω being one of the two values differing by $\Delta\omega$. For example, if $R=0.5$, then $2\gamma=\Delta\omega$ $=$FWHM as expected.

Since $\Delta\omega$ and R are easy to measure directly from the experimental curve, a number of different "measurements" of γ can be made by selecting different values of R. This generally produces less error and also yields an uncertainty via the standard deviation of the data. It also serves as a check that the curve is indeed the Lorentzian given by (1). As an example, consider the following data taken from Ref. 2:

A	R	$\Delta\omega$	γ	$Q=\omega_0/2\gamma$
8	...	...	...	...
7	0.766	66.6	60.2	11.6
6	0.563	122.8	69.7	10.0
5	0.391	159.3	63.8	10.9
4	0.25	230.6	66.6	10.5
3	0.141	343.4	69.5	10.0
2	0.063	609.5	78.7	8.9

Here, A is the amplitude of the oscillation as measured from the oscilloscope, which displays $x(t)=A\cos(\omega t+\phi)$. As shown in Ref. 1, $A^2(\omega)$ is approximately a Lorentzian. At resonance, A was 8 divisions on the oscilloscope screen, so we take $R=(A/8)^2$. The values of $\Delta\omega$ shown above were measured using a frequency counter, which also measured the peak frequency to be 222 Hz, or $\omega_0=1395\ \text{s}^{-1}$. The values of γ were calculated using (2) for each value of R and $\Delta\omega$. The purpose of the experiment was to find $Q=\omega_0/2\gamma$, which is also listed.

From the data we see that a systematic error is becoming significant when $R<0.2$. This systematic error is due to at least three complexities: First, the measured amplitude contained an unwanted background sound. Second, the driving force F_0 had a slight frequency dependence, which we are neglecting here. And third, the function to which we are trying to fit is not really a true Lorentzian, but instead obeys,

$$L_1(\omega)=\frac{4\omega_0^2}{(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2}. \tag{3}$$

If we take only the first four data points in order to suppress these systematic errors, we obtain $\gamma=65\pm4\ \text{s}^{-1}$ and $Q=10.8\pm0.7$.

We now consider complications that arise from the fact the response curve of a driven harmonic oscillator is not exactly a Lorentzian. Equation (3) becomes a true Lorentzian in the limit $\gamma\ll\omega_0$. To correct for this effect, multiply by the correction term:

$$\gamma=\gamma_0\left(1-\frac{\gamma_0^2}{2R\omega_0^2}+\cdots\right)=\gamma_0\left(1-\frac{1}{8RQ^2}+\cdots\right), \tag{4}$$

where γ_0 is the width obtained using Eq. (2), and γ is the parameter that appears in the ordinary differential equation for a damped driven harmonic oscillator:

$$m\ddot{x}=-kx-2m\gamma\dot{x}+F_0\cos(\omega t). \tag{5}$$

As an example of how to use (4), consider the measurement associated with $R=0.063$. Using $Q=10.8$, the correction term in (4) is seen to be 0.983. This implies that the experimental value of $\gamma=78.7\ \text{s}^{-1}$ (for $R=0.063$) should be replaced by $\gamma=(78.7)(0.983)=77.4\ \text{s}^{-1}$. This brings γ slightly closer to the established value of 65 s^{-1}, but the correction term appears to play an insignificant role for this experiment. We conclude that the other sources of systematic error mentioned above are dominant. The correction indicated by (4) is smaller than 1% for the other values of R, so that the approximation (2) is clearly adequate for our purposes.

Finally, we outline the lengthy steps leading to (4): Putting the non-Lorentzian response of a harmonic oscillator into dimensionless variables $x=\omega/\omega_0$ and $\beta=\gamma/\omega_0$, one obtains

$$x=\left[1+\left(-2\beta^2\pm2\beta\sqrt{\frac{1-R}{R}(1-\beta^2)}\right)\right]^{1/2}, \qquad (6)$$

for the two frequencies where the response falls to a factor of R below its peak value. This can be expanded to third order in the small parameter β. The expansion is accomplished by expanding the two nested Taylor expansions $(1+\epsilon)^{1/2}$, there ϵ is either β^2 or the term appearing in the large parenthesis ( ) of (6). Defining $\Delta x=\Delta\omega/\omega_0$, one obtains an equation involving Δx that is a third order polynomial in β. Equation (4) follows from an approximate solution of that polynomial.

[1]Grant Fowles, *Analytical Mechanics* (CBS College Publishing, New York, 1986), 4th ed., Sections 3.3 and 3.4.

[2]G. Vandegrift, "Experimental study of the Helmholtz resonance of a violin," Am. J. Phys. **61**, 415–421 (1993).

THE LUMINIFEROUS ETHER

We must not listen to any suggestion that we must look upon the luminiferous ether as an ideal way of putting the thing. A real matter between us and remotest stars I believe there is, and that light consists of real motions of that matter—motions just such as are described by Fresnel and Young, motions in the way of transverse vibrations. If I knew what the magnetic theory of light is, I might be able to think of it in relation to the fundamental principles of the wave theory of light. But it seems to me that it is rather a backward step from an absolutely definite mechanical motion that is put before us by Fresnel and his followers to take up the so-called electro-magnetic theory of light in the way it has been taken up by several writers of late. In passing I may say that the one thing about it that seems intelligible to me, I scarcely think is admissible. What I mean is, that there should be an electric displacement perpendicular to the line of propagation and a magnetic disturbance perpendicular to both. It seems to me that when we have an electro-magnetic theory of light, we shall see electric displacement as in the direction of propagation—simple vibrations as described by Fresnel with lines of vibration perpendicular to the line of propagation—for the motion actually constituting light. I merely say that in passing, as perhaps some apology is necessary for my insisting upon the plain matter of fact dynamics and the true elastic solid as giving what seems to me the only tenable foundation for the wave theory of light in the present state of our knowledge.

The luminiferous ether we must imagine to be a substance which so far as luminiferous vibrations are concerned moves as if it were an elastic solid. I do not say it is an elastic solid. That it moves as if it were an elastic solid in respect to the luminiferous vibrations is the fundamental assumption of the wave theory of light.

William Thomson (Lord Kelvin), 1884. *Kelvin's Baltimore Lectures and Modern Theoretical Physics*, edited by Robert Kargon and Peter Achinstein (MIT, Cambridge, 1987), p. 12.