LETTERS TO THE EDITOR

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A simple derivation of the Green's function for a rectangular Helmholtz resonator at low frequency

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The Green's function for a Helmholtz resonator is obtained by considering a closed box containing a point source radiator. Using the method of images, a low-frequency approximation is derived for the special case of a rectangular box.

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Here is a simple derivation of the Green's function for a resonant cavity in the limit of low frequency, apparently first derived by Van Bladel.[1] Consider a point radiator located at $\mathbf{r}'$, near the center of a closed rectangular box. The pressure at point $\mathbf{r}$ obeys

$$\nabla^2 P(\mathbf{r}) + k^2 P(\mathbf{r}) = Q\delta^3(\mathbf{r}-\mathbf{r}'),$$

where $k=\omega/c_s$ is the wavenumber and Q is the strength of the source. The boundary condition is tht $\nabla P\cdot\mathbf{n}$ vanishes at the surface of the box, where $\mathbf{n}$ is the outward unit normal. Using the method of images, the walls of the box can be replaced by the image sources obtained by reflecting the source point at $\mathbf{r}'$ (and all its images) about the walls. There is an infinity of such image sources. Assuming that pressure oscillates as the real part of $e^{j\omega t}$, we have the exact expression

$$P(r) = \frac{-Q}{4\pi} \sum_i \frac{\exp(-jk|\mathbf{r}-\mathbf{r}_i|)}{|\mathbf{r}-\mathbf{r}_i|},$$

where $\mathbf{r}_i$ is at the source or at one of its images. The approximation, valid at low frequency, consists of treating the source term inside the box separately, and using an integral to represent the sum over all the images,

$$P(r) \approx \frac{-Q}{4\pi}\left(\frac{\exp(-jk|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} + \int \frac{e^{-jkr}}{r}\frac{d^3\mathbf{r}}{V}\right),$$

where V is the volume of the box, and we note that the average density of image sources is V^{-1}. The first term represents the contribution of the source at $\mathbf{r}=\mathbf{r}'$, which is generally small, as will be shown later. This justifies our treatment of the nearby images as part of the integral because these nearby images are all outside the box, and are important only if the original source is very close to a wall.

Two conditions must be met in order to ensure that the integral, taken over all space, is a reasonable approximation for the sum over images. First, the phase shift between neighboring images must be small, allowing us to treat image points as a continuous source. This requires that $kL \ll 1$, where L is a dimension of the box. Thus, our Green's function would not be useful for a resonator that resembled a long pipe, which supports standing waves.

A second condition for allowing us to replace the sum by an integral rises from the fact that the integral approximation works best for the distant images. The relatively small number of images close to the box should not contribute as much to the integral as do the large number of distant images. This can be verified by noting that the integrand (with $d^3\mathbf{r}=4\pi r^2\,dr$) diverges for large r. Performing the integration yields[2]

$$P(\mathbf{r}) \approx \frac{-Q}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \frac{Q}{k^2 V},$$

where in the first term we have approximated the exponential as unity, since $k|\mathbf{r}-\mathbf{r}'| \ll 1$. If the source is at a wall or corner, there will be a cluster of images at nearly the same location, so that one should multiply the first term by 2, 4, or even 8 (if the source is located at the corner).

The second term Q/k^2V results from the integration over all the images. The ratio of this term to the first term can be simplified using the formula $k^2=2R/V$ for the resonant frequency of a Helmholtz resonator[3] with a small circular aperture of radius R:

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$$\frac{4\pi|\mathbf{r}-\mathbf{r}'|}{k^2V}=2\pi\frac{|\mathbf{r}-\mathbf{r}'|}{R}.$$

This parameter must be small for typical values of $|\mathbf{r} - \mathbf{r}'|$, for the following reason. If the parameter were not small we would not have been justified in treating the nearby images as part of the integral. In other words, if the source term is important for moderately large $|\mathbf{r}-\mathbf{r}'|$, then some of the nearby image terms will also be important. We see that the source term is not important unless one is within a distance R of the source. Thus our approximation is useful for a Helmholtz resonator with an aperture size that is much smaller than the dimensions of the box: $R \ll V^{1/3}$. As with the previous approximation ($kL \ll 1$), this is a low-frequency approximation.

To obtain the pressure inside an actual Helmholtz resonator, we represent the aperture as a surface acoustical charge density, and note that the above expression for $P(r)$ is a Green's function.[4] We insert a factor of 2 to include the image on the other side of the wall (the aperture is assumed not to be at a corner):

$$P(r)\approx\int\left\{\frac{1}{k^2V}-\frac{2}{4\pi|\mathbf{r}-\mathbf{r}'|}\right\}\sigma(\mathbf{r}')d^2\mathbf{r}',$$

where the integral is over the surface of the aperture. By analogy with electrostatics, the surface charge σ is proportional to the normal component of the velocity,

$$\sigma=\nabla P\cdot\mathbf{n}=-j\omega\rho(\mathbf{v}\cdot\mathbf{n}),$$

where ρ is the mass density of air.

In conclusion, we have derived a Green's function that describes a Helmholtz resonator with an aperture that is much smaller than the dimensions of the box. We also require that the box not be so long and thin that the condition $kL \ll 1$ is violated.

[1] J. Van Bladel, "Coupling through Small Apertures, with an Application to Helmholtz' Resonator," J. Acoust. Soc. Am. **45** 604–613 (1969).

[2] This integral is meaningful only if k has an arbitrarily small but nonzero damping term, which causes the contribution to the integral from images at infinity to vanish. This can be justified by noting that some intensity will be lost due to multiple reflections from the walls. Or one can assume that the source is slowly turned on, so that the lookback time to the distance sources ensures that their contribution weakens exponentially with distance. This is reminiscent of Olber's paradox in cosmology. See, for example W. K. Hartman, *Astronomy: The Cosmic Journey* (Wadsworth, Belmont, CA, 1989), p. 587.

[3] L. E. Kinsler and A. R. Frey, *Fundamentals of Acoustics* (Wiley, New York, 1958), p. 218.

[4] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), p. 224.