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# Experimental study of the Helmholtz resonance of a violin

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The violin supports a Helmholtz resonance, acting like a driven, damped harmonic oscillator. The amount of damping can be measured from the width of the response curve, and also from the decay of an undriven oscillator. The phase shift between driver and resonator is also consistent with theory. Shifts in resonant frequency due to changing the aspect ratio of the *f* hole are measured and found to be in qualitative agreement with a convenient formula for the resonant frequency associated with a long thin aperture.

## I. INTRODUCTION

> "The suppositions on which we are about to proceed are not of course strictly correct as applied to actual resonators such as are used in experiment, but they are near enough to the mark to afford an instructive view of the subject and in many cases a foundation for a sufficiently accurate calculation of the pitch."
>
> From *The Theory of Sound* by Lord Rayleigh, first published in 1878

I was looking for an interesting laboratory for a waves course when I obtained a cheap violin. Its investigation resulted in an advanced physics lab, a demonstration on sound and oscilloscopes for junior high school students, and insights into how the shape of a violin's *f* hole determines the resonant frequency. And, while reading about Helmholtz resonators in an old book, I discovered the above passage.[1]

Put a coffee cup to your ear, leaving a small gap between your head and the cup, and listen to the "sound of the ocean." You will notice an increase in pitch as you increase the size of the gap. Equation (3a) below predicts such a relationship between pitch and gap size, which suggests that a Helmholtz resonance is being excited by the background sounds. The original Helmholtz resonator was a hollow sphere with one or two small cylindrical tubes connecting the inside of the sphere to the open air. It was investigated by Herman von Helmholtz, and then by Lord Rayleigh, who calculated the resonant frequency of a large box with an elliptical aperture. As a Helmholtz resonator oscillates, air passes through the aperture, causing the internal pressure to oscillate.

The physics of the violin had been extensively studied in order to understand how violins work[2–4] and to create new designs.[5,6] An isolated violin string cannot radiate much sound for the same reason a pencil would make a poor fan on a hot summer day. The surface area is too small to move an appreciable amount of air. By exciting various normal modes of the violin, the string is able to set a larger object in motion, and hence move more air. For notes near what violinists call "open *D*" (293 Hz) a mode closely resembling the Helmholtz resonance becomes important.[3,4,7–13] This mode is not a true Helmholtz resonance because some motion of the walls is involved.[3,8,9] It has been called the "*f*-hole resonance," the "$A_0$ resonance," and the "main air resonance." An accomplished violinist, unaware of all this physics, simply calls it "a sweet note." Among the well-known modes, the main air mode has the lowest frequency.[7]

In this paper, we show how to demonstrate a fundamental system in physics: a driven, damped harmonic oscillator.[14,15] The experiment consists of a small speaker placed over one of the *f* holes, with a microphone over the other. Two different methods for measuring the damping term agree to within the experimental uncertainty of less than 10%. We also investigate a simple formula for a long, thin aperture that shows how resonant frequency depends more

  

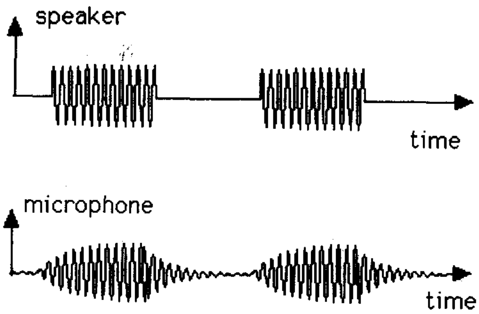


Fig. 1. Depiction of signals to the speaker (top) and from the microphone (bottom) when driving the speaker with "tone bursts," The undriven decay of the resonator is the exponentially decaying signal at the tail end of the pulse (bottom).

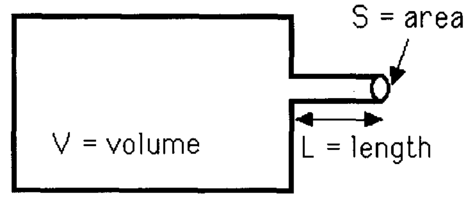


Fig. 2. A Helmholtz resonator consisting of a box of volume $V$, and a tube of length $L$, and area $S$.

strongly on the $f$-hole's length than on its width. This might be of interest to designers of stringed instruments.

To measure the damping term by the method of "undriven decay," the speaker is driven by a sine wave generator that can be pulsed off in order to produce the "tone bursts" shown in the top part of Fig. 1. As has been demonstrated with closed acoustical cavities,[16] the undriven acoustical signal shows an exponential decay in amplitude, as seen in the trailing edge at the bottom of Fig. 1. In the method of "driven response," the speaker is driven by a steady sine wave, and the amplitude from the microphone is measured as a function of driving frequency. The damping term is obtained from the width of this response curve. A third, albeit crude, measurement of the damping term is made from the phase shift between the speaker and the microphone.

As a supplement to this experiment, it is worthwhile to invite a violinist to demonstrate the air resonance on a good violin. Coach the violinist to find the air resonance by playing a slow glissando on the $G$ string and listening for a hollow sound. Our violinist had difficulty finding it because she unconsciously used less bow pressure at resonance. We covered[17] one $f$ hole and asked her to play scales. As explained below, this shifts the resonant frequency lower by a factor close to $\sqrt{2}$. The resonance was immediately discernible at the new frequency, but the violinist rapidly learned to compensate, and the resonance was again almost hidden.

## II. CALCULATION OF THE RESONANT FREQUENCY

We start with a derivation based on simple geometry that only crudely describes a violin.[1,3,4,12] It is reviewed here because it should be part of the physics lab and because it will lead to new insights about air flow through the $f$ holes. An enclosed box of volume $V$ has a long cylindrical opening of length $L$ and cross-sectional area $S$, as shown in Fig. 2. We assume

$$S^{1/2} \ll L \ll V^{1/3} \ll \lambda \tag{1}$$

where $\lambda = 2\pi c/\omega$ is the "free wavelength of sound" and $c = (\gamma P/\rho)^{1/2}$ is the speed of sound. The system can be pictured as a mass $m$ and a spring constant $k$. The mass of air inside the tube is $m = \rho SL$, where $\rho$ is the mass density of air. As the Helmholtz resonator oscillates, this air undergoes a very small displacement $x(t)$ within the tube.

To obtain the resonator's spring constant $k$, we take the force on the air in the tube to be $F = -kx$. Defining $\delta P$ as the pressure difference across the tube, we have $F = S\,\delta P$. Taking the differential of the adiabatic law ($PV^\gamma = \text{const.}$) yields $\delta P/P = -\gamma(\delta V/V)$. Using $\delta V = Sx$, one obtains for the spring constant of air,[15]

$$k = \frac{\gamma P S^2}{V}. \tag{2}$$

The resonant frequency is

$$\omega_0^2 = \frac{k}{m} = \frac{c^2 S}{LV}, \tag{3a}$$

$$\omega_0^2 \approx \frac{c^2 a}{2V}. \tag{3b}$$

The approximation (3b) is obtained by first setting $S = \pi ab/4$ as the area of an ellipse of length $a$ and width $b$, and then using the fact that $L$ is typically close to $1.8b$ for a violin. In this approximation the resonant frequency depends only on the length of the aperture.

A better approximation, which is a handy modification of Lord Rayleigh's formula, is

$$\omega_0^2 \approx \frac{c^2 a'}{V} \frac{\pi}{2 \ln(4a/b)}, \tag{4}$$

where $a'$ is the sum of the lengths of both slits, and $a$ is the length of one slit. (This assumes that the slits are so far apart that they do not interact with each other, which is only an approximation.) Comparing (3a) and (4), we see that the effective length is evidently

$$L \approx \frac{b}{2} \ln\left(\frac{4a}{b}\right). \tag{5}$$

For $a/b = 10$, we have $L = 1.84b$ and $(\pi b/4L) = 0.43$, which is close to the simple factor $\frac{1}{2}$ assumed in (3b).

To understand this formula for the effective length, we use an analogy with the theory of electrostatic fields.[1,3,18] The air velocity associated with the resonator is of the form $\mathbf{v}(\mathbf{r})\cos(\omega t)$, where both the curl and divergence of $\mathbf{v}(\mathbf{r})$ nearly vanish. The problem of finding the resonant frequency of an aperture is equivalent to that of finding the electric conductance of the same aperture if it is cut out of an insulator and immersed in a conducting fluid. It is also equivalent to finding the electrostatic capacitance of a conducting sheet with the aperture's shape. If the slit of the resonator is long and thin, and wall thickness is neglected, we can exploit the approximate two-dimensional nature of the geometry, and use a conformal map[18] in the complex plane to describe the air flow near the middle of the slit, as shown in Fig. 3.

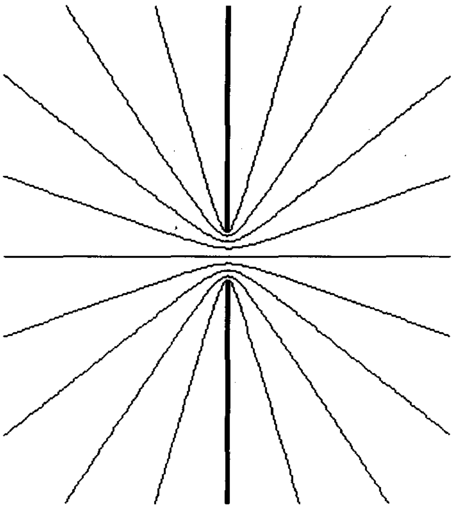

Fig. 3. The flow lines of $\mathbf{v}$ for an incompressible fluid flowing through a infinitely long slit. The width of gap represents the width, $b$, of the slit. The slit length, $a$, is taken to be infinity. The direction of $\mathbf{v}$ is parallel to the flow lines with the magnitude of $\mathbf{v}$ being inversely proportional to the distance between field lines. The contour map is $Z=\cos(iW)$.

Lord Rayleigh[1] used a superior geometry and considered an ellipsoid. By taking the limit as one dimension vanishes, he derived the capacitance of an elliptical sheet of length $a$ and width $b$. His result could only be expressed in terms of an elliptical integral. But, if the slit is long and thin, $(a \gg b)$, we can use the Taylor expansion,[19]

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-(1-\epsilon)^2\sin^2(\phi)}} = \ln\frac{4}{\epsilon} + \frac{1}{2}\left(\ln\frac{4}{\epsilon} - 1\right)\epsilon^2 + \cdots . \tag{6}$$

For an ellipse of arbitrary aspect ratio, replace $\ln(4a/b)$ in (4) and (5) by the left-hand side of (6) with $\epsilon = a/b$.

Looking at Fig. 3, one might not be surprised that $L \approx 2b$ because air speed is largest within a distance $b$ of the slit. It is easy to image a "tube" of length $b$ extending on both sides of the aperture. We shall now see that this should not be taken too literally, especially in the limit $a \gg b$.

Insight into the spatial extent of air flow around the $f$ hole is obtained by exploiting a fundamental relationship between resonant frequency and the kinetic and potential energy.[15] To obtain the kinetic energy, we need to integrate $\int v^2\, d\tau$ over all space. From Fig. 3, $v \propto r^{-1}$ for $r \gg b$, so that $\int v^2\, d\tau$ diverges as $r$ goes to infinity. The divergence is removed because the two-dimensional conformal map fails to describe air flow when $r \gg a$. This is analogous to the electric field caused by a charged wire of length $a$: the field falls as $r^{-1}$ if $r \ll a$ and as $r^{-2}$ if $r \gg a$. The logarithms in (4) and (5) are essentially truncations of a logarithmic divergence at $r \approx a$.

From this analysis, we see that for a long thin aperture $(a \gg b)$, the most significant contribution to the kinetic energy comes from a region extending from $r \approx b$ out to $r \approx a$. The sphere of influence extends outward from the aperture almost as far as it is long. This can be verified by observing small changes in the resonance as objects are brought near an $f$ hole. One must be careful to keep the setup in front of the $f$ hole the same if one wants repeatable measurements. Since $\rho\, \partial\mathbf{v}/\partial t = -\nabla P$, air flow near the $f$ hole is closely related to the nonuniformity of perturbed pressure, discussed in a recent paper by Shaw.[11] The proximity of the $f$ holes to each other and the back plate ensures that (4) will not be what Lord Rayleigh calls "strictly correct," because of the air flow is modified.

Aside from inadequate compliance with inequality (1), there are other reasons (4) is not strictly correct. Cremer[3] calculates an 8% decrease in resonant frequency due to the finite thickness of the plate, which can be modeled as an increase in mass $m$. He also estimates a shift of about 5% due to the flexibility of the walls, which is also discussed by Hutchins,[10] and by Jansson.[8] Not only does the bizarre shape of an $f$ hole defy rigorous calculation, but the volume of a violin is difficult to measure. Using a ruler, I estimated the volume to be about 2400 cm$^3$, which is 10% larger than the value used by Cremer, and 30% larger than claimed in Itokawa. Since resonant frequency is proportional to the square root of volume, these discrepancies in volume lead to changes in calculated frequency of about 5% and 14%, respectively.

We see that a large number of complications make it very difficult to calculate the resonant frequency to precision better than 10%. What makes this lab interesting is not the precision of the measurements, but the number of ways one can verify the proposition that the air resonance of a violin resembles a driven, damped harmonic oscillator.

## III. THE DAMPED, DRIVEN OSCILLATOR

The driven, damped harmonic oscillator obeys a well-known linear differential equation.[14,15,20] The dimensionless "quality factor" $Q$ is

$$Q \approx \frac{\omega_0}{2\gamma}, \tag{7}$$

where $\gamma$ is the drag term in Eq. (5) of Ref. 20. Since the amplitude $x(t)$ of an undriven oscillator decays as $e^{-\gamma t}$ for large $Q$, we can use (7) to deduce $Q$ from observation of the undriven decay shown in Fig. 1.

We can also deduce $Q$ from the driven response of the resonator by varying the driving frequency and measuring the driven response. If the driving force is $F_0\cos(\omega t)$, then the displacement obeys $x(t) = A\cos(\omega t + \phi)$, where $A = A(\omega)$ is a function of the driving frequency. As shown in Ref. 20, $A^2$ is nearly a Lorentzian curve with a peak at $\omega \approx \omega_0$. The width of this curve is related to $Q$ by

$$Q \approx \frac{\omega_0}{\text{FWHM}}, \tag{8}$$

where FWHM (full width at half maximum) is the difference between the two frequencies for which $A^2$ falls to half its peak value. From (7) and (8), we see that the FWHM of $A^2$ is twice the damping constant: FWHM$=2\gamma$.

The phase shift of $x(t)$ with respect to the driving force $\phi$ is given by

$$\phi = \arctan\left(\frac{2\omega\gamma}{\omega_0^2 - \omega^2}\right) = \arctan\left(\frac{\omega\omega_0 Q^{-1}}{\omega_0^2 - \omega^2}\right), \tag{9}$$

so that one could also measure $Q$ by observing how $\phi$ varies with $\omega$. Theoretical and experimental graphs of $A^2$ and $\phi$ vs $\omega$ are shown in Figs. 4 and 5.

Since (7)–(9) represent three independent ways to measure $Q$, the hypothesis that the air mode resembles a driven

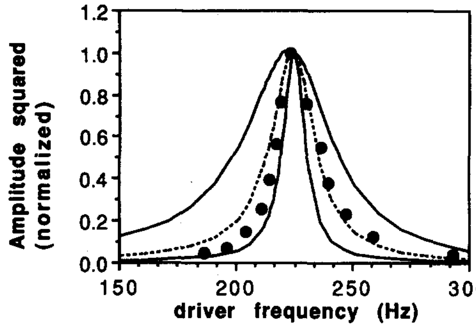


Fig. 4. Comparison of theoretical and experimental response curves. Measurements of $A^2(f)$ are shown along with theoretical curves for $Q=5$, 10, and 15. The curves are each normalized so that they have the same maximum response. In fact, the maximum response is greater for larger values of $Q$. The experimental data was multiplied by $(1-0.05\Delta f)$ to account for the frequency dependence response of the speaker. (Here $\Delta f = f - f_0$). The theoretical curves are shifted slightly to the right in order to better fit the data.

oscillator can be tested by measuring the three values of $Q$ and showing them to be equal to within experimental error.

## IV. EQUIPMENT

The equipment necessary for this lab (demonstration) is (1) A dual trace oscilloscope. (A storage oscilloscope is preferable.) (2) A microphone with sufficient output to permit observation on the oscilloscope. (3) A gatable sine wave generator that can drive a small speaker. (4) A full-sized violin or a box of comparable dimensions with slits. (5) A frequency counter (optional but highly recommended).

The sound from the speaker should be clearly audible at frequencies down to about 150 to 200 Hz. Speakers with diameters between 1 and 3 in. are readily available, usually have impedances of 8 Ω, requiring a 20 V-signal (peak to peak) in order to produce an acceptable loudness. These small speakers exhibit an annoying dependence of amplitude with frequency, but the experiment succeeds due to the sharpness of the resonance. A larger and more powerful speaker is suggested for classroom demonstrations. (One can also excite the resonance by singing into an $f$ hole.)

The gatable sine wave generator needs to produce tone bursts of about 60 ms duration as shown in the top part of Fig. 1. We used a Wavetek model 188 sweep/function generator that cost $980, but any wave generator would work if you are willing to build a small gating circuit and amplifier. The violin need not cost over $50, as it will be handled and probably mauled slightly. It can be of truly bad quality.

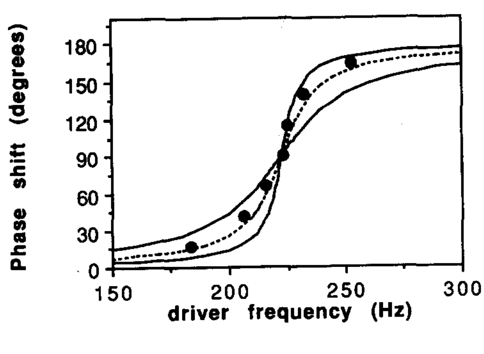


Fig. 5. Measurements of the phase shift between the driver and resonator are shown along with theoretical curves for $Q=5$, 10, and 15. The observed data points were all shifted by a constant phase because the absolute phase shift was not measured here.

## V. COMPARISON OF DIFFERENT METHODS FOR MEASURING $Q$

The goal was to achieve the maximum possible agreement between two different methods of measuring $Q$ ("driven response" and "undriven decay"). For this reason, the setup remained the same for each measurement. An important cause of discrepancy between the two methods is that the method of driven response suffers from a background of signal reaching the microphone directly from the speaker. In order to reduce this unwanted signal to a minimum, both the speaker and microphone were as close as possible to the $f$ holes.

A small (1 in.) speaker was taped over one $f$ hole. A piece of masking tape was taped around the perimeter of the speaker, so that the speaker was 1–2 mm from the violin when placed face down on it. This was to prevent the speaker from causing the wood to vibrate and unwanted sound coming directly from the speaker to the microphone. The remaining part of the $f$ hole with the speaker was also covered with tape, so that the violin had only one open $f$ hole. The sides of the microphone were also covered with tape to make it as directional as possible. The end of the microphone was placed about one millimeter from the other $f$ hole.

As a test of whether the sound was successfully blocked from the speaker to the microphone, a piece of paper was placed over the remaining $f$ hole with the speaker set at the resonant frequency. The amplitude from the microphone fell by a factor of approximately 14 as the resonant frequency was shifted far from the frequency of the speaker.

The measurement of $Q$ by undamped decay proceeded as follows: The Wavetek signal generator was set to produce "tone bursts" with a 50% duty cycle by triggering from a 8 Hz square wave of another generator. Both the signal to the speaker and the signal from the microphone were simultaneously displayed on a dual trace oscilloscope, as depicted in Fig. 1. When taking measurements, the oscilloscope trace was set so that zero signal corresponded to the bottom of the screen, and the driven amplitude was set to almost fill the screen of the oscilloscope, which showed a series of peaks with decreasing amplitude after the generator was shut off. This allowed for fairly precise measurements of the maximum amplitude directly off the scope.

The logarithm of the amplitude of each peak versus time is plotted in Fig. 6. Linear regression yields a slope $\gamma = 62.9$ $s^{-1}$. Taking the resonant frequency to be $\omega/2\pi = 223$ $s^{-1}$, we obtain $Q = 11.1 \pm 0.8$ using (7). The uncertainty is taken as the standard deviation of values of $Q$ obtained by comparing each subsequent peak with the first, so that each peak after the first constitutes a "measurement" of $Q$.

The amplitude of the speaker (as measured by the mi-

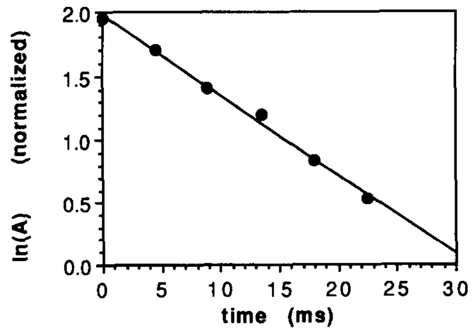


Fig. 6. The logarithm of the amplitude of the undriven signal versus time. Each point represents the local maximum ("peak") in the microphone signal after the speaker was shut off.

crophone) increases in amplitude by about 10% for each increase in frequency of 20 Hz. Figure 4 shows a plot of experimental data points after this effect was removed by multiplying the observed amplitude by a correction factor. Also shown are theoretical curves for $Q=5$, 10, and 15. Using the method of Ref. 20 to deduce the width of experimental curve, one obtains $Q=10.8\pm0.7$. The discrepancy between both methods of measuring $Q$ is within experimental error.

## VI. PHASE SHIFT OF DRIVEN RESPONSE

The theoretical phase shift between the driver and resonator given is plotted in Fig. 5 for three values of $Q$. In principle, this is a third way to measure $Q$, but one can only conclude that $Q$ is between 10 and 14, which is consistent with the more precise measurements reported above.

We now consider the absolute phase shift between the driver and the resonator. The method of obtaining the phase shift shown as the data of Fig. 5 only measures the change in phase as one sweeps through resonance. This is because the difference in phase between the voltage to the speaker and the pressure at the $f$ hole is unknown. (This difference was presumed to be unchanged as one sweeps through resonance.) In order to measure the absolute phase shift, the oscilloscope was triggered externally by the voltage to the speaker and the phase shift in the microphone signal was measured at resonance. Keeping the driving frequency constant, the $f$ hole was partially covered. The microphone signal became much weaker and shifted in phase by 110° as the system left resonance. Comparing this with the expected phase shift of almost 90°, we conclude that the discrepancy between theory and experiment is 20/180, or 11%.

## VII. $Q$ OF AN ISOLATED VIOLIN

Because wall motion is involved, the value of $Q$ probably depends slightly on how the violin is excited. Our setup only permits acoustical excitation of the air resonance. But we can try to determine $Q$ when both $f$ holes are open and reasonably isolated from the speaker. The previous setup was designed to obtain the same values of $Q$ using the methods of "undriven decay" and "driven response." Here we try to find $Q$ of an isolated violin when the air resonance is excited acoustically.

A larger (6 cm) speaker was placed 4.5 cm away from one $f$ hole. This makes the measurement by "driven response" impractical because of unwanted sound from the speaker to the microphone. The only problem with the method of "undriven decay" is that sound bouncing off the walls of the room will imitate the decaying Helmholtz oscillation. However, the degree to which reverberation is important can be investigated by covering the $f$ holes, which destroys the resonance. Under these circumstances, only the reverberation signal appears, which was 16 times smaller than the amplitude of the air resonance. This reverberation could be reduced to an unobservable value by exciting the resonance with the speaker on the back surface of the violin. Measurements of $Q$ yielded $14\pm1$ with the speaker in front of the $f$ hole, and $15\pm1$ with the speaker behind the violin. Jansson[8] has measured $Q$ values for a number of air resonances. For the lowest-order mode of interest to us, his graph indicates $Q=14.5$, so that our measurements are in agreement with that claimed by Jansson.

A finite value of $Q$ is associated with the energy loss from the resonator. Cremer[3] claims that the most important loss mechanisms are radiation of sound waves and viscous heating of the air as it rushes through the holes. My tentative calculation yields $Q=19$, with large uncertainties, so that $Q$ in the range 16–22 could be reasonably calculated. Taking the observed value of $Q$ to be 14.5, I calculate that 47% of the energy is lost as sound radiation, 18% as viscous heating, 7% as flexing of the wood plates, and 2% from heat loss during the near adiabatic compression (decompression). This leaves 26% of the observed energy loss unaccounted for. The fit between theoretical and experimental values of $Q$ is marginal.

## VIII. SHIFT IN RESONANT FREQUENCY DUE TO CHANGES IN $f$ HOLES

The following parameters were taken for the violin: volume $V=2400\ \text{cm}^3$, slit length $a=8.5$ cm, and slit width $b=0.5$ cm. Using (4) with $c=3.4\times10^4$ cm/s, we obtain a theoretical resonant frequency $\omega/2\pi=196$ Hz, which is 11% below the observed 223 Hz. The resonance with both $f$ holes open was observed to be 288 Hz. The observed ratio of $288/223\approx1.29$ is 9% lower than the theoretical ratio of $\sqrt{2}\approx1.41$ for the ratio of resonant frequency with two holes to that with one hole open.

The parametric dependence of resonant frequency on the slit dimensions $(a,b)$ was investigated by first taping over the ends of the $f$ hole until a rectangular aperture remained of dimensions $5\times0.55$ cm ($\omega_0/2\pi=185$ Hz). With the hole reduced to a rectangle, its aspect ratio could be varied by taping over a small sections of it, reducing the size of either $a$ or $b$. Figure 7 shows the effect of continuously making the rectangle shorter and also of making it more narrow. The two lines show the theoretical curves based on (4). In both the experimental and theoretical curves, the variation with the least frequency dependence corresponds to reducing the slit width. As noted above, we do not expect exact agreement between theory and experiment due to a number of complications. However, there is good ex-

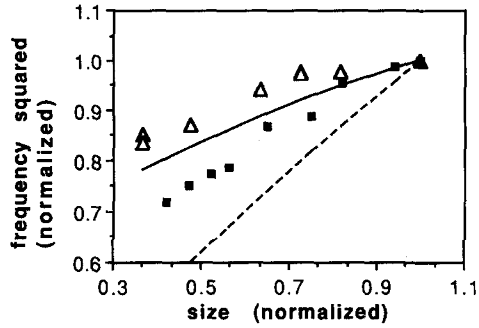


Fig. 7. Variation of resonant frequency with aspect ratio of the aperture. The starting point was a rectangle of dimensions 5×0.55 cm, which represents the highest frequency and largest size shown in the upper right corner. The size (either thickness or length) was then reduced by the factor indicated in the graph, which caused the resonant frequency to drop. The open triangles represent the shift in frequency squared when the aperture was made more narrow. The solid squares correspond to making the aperture shorter. The corresponding theoretical curves are shown, with the solid and dotted lines representing decreases in width and length, respectively.

perimental evidence that it is acceptable to use masking tape to form the apertures, as is shown next.

An experiment was carried out to investigate the difference between forming the aperture of masking tape and of a 0.2 cm aluminum plate. A matched pair of aluminum plates of dimensions 2.5 cm×5.0 cm was prepared. On one of the plates, about 40%, was cut away and replaced with masking tape. Clay was molded around the *f* hole to form a jig that could hold either plate, allowing one to interchange plates without changing the plate's position. To reduce random error associated with imprecise placement of the plate, the resonant frequency was measured many times in succession, switching plates after each measurement. The most repeatable method to find the resonant frequency is to take the average of two frequencies where the amplitude falls to about 95% of its maximum value.

The experiment was carried out under two conditions. First the aperture's length was reduced to about half that of a normal *f* hole, while the width was unchanged. The difference between resonant frequencies with masking tape and aluminum masks was $0.4\pm0.2$ Hz, or 0.16%. The low uncertainty was obtained by repeating the measurement 17 times and taking the standard deviation of the mean. The equality between masking tape and aluminum is so perfect that I am puzzled. A second experiment to make the aperture more narrow at the same length was less repeatable due to difficulties in placing the mask over the curved surface of the violin. Here, the change in frequency between aluminum and tape mask was no more than 0.3%. However, both experiments indicate that the data of Fig. 7 is probably reliable, in spite of the fact that masking tape was used to form the apertures.

## IX. CONCLUSION

This laboratory has been incorporated into a project for physics majors that starts with a brief history of the violin,[2] and then proceeds to a guided derivation of (3a) starting from Fig. 2. The students then obtain the value of "$Q$" using the methods of "undriven decay" and "driven response."

One merit of this investigation is the way different areas of physics are brought together. Ideas from *mechanics, thermodynamics,* and *electrostatics,* as taught in the first two years of college, have been used here. At a slightly more advanced level, one can follow Cremer[3] and model the flexing of the violin's walls as a system of coupled harmonic oscillators. For those with advanced *electrodynamics,* one can talk about sound radiation, which is the primary energy loss mechanism in this experiment. Cremer[3] gives an introduction to acoustical radiation theory, which is much simpler than electrodynamic theory, yet involves the concepts of retarted potentials, energy flux, and near versus far-field radiation. Here, the acoustic analog for the electromagnetic wave equations is $c^2\nabla^2P=\partial^2P/\partial t^2$ for low-amplitude pressure fluctuations.

At the other extreme of student sophistication, I have shown this lab to eighth grade students, where the emphasis is on how an oscilloscope displays sound vibrations. The driven Helmholtz resonator is explained to be analogous to a heavy person on a swing being pushed by a small child. The students hear the tone bursts from the speaker, and are then asked to imagine what would happen if the small child pushed the heavy person on the swing for a minute and then rested. The appearance of Fig. 1 on the oscilloscope makes a nice conclusion.


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# A Galilean experiment to measure a fractal dimension

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A simple and pedagogical kinematic experiment to measure the mass-size fractal dimension $D$ of crumpled surfaces is presented. The experiment is inspired in the well-known Galilean experiments with bodies rolling on inclined planes. The experimental results for $D$ agree with those obtained from static means.

The problem of a hoop, a disk, or a sphere of radius $R$ rolling from rest down a ramp is discussed in any introductory physics course.[1] In this article, we revisit this problem using a new kind of rolling object, namely the (non-Euclidean) fractal crumpled surface (CS).[2] CS are obtained from random and irreversible compaction of paper sheets or aluminum foils.[3] These fractal systems present novel and interesting critical behavior as discussed recently.[4]

Here, we present an experimental and pedagogical method to obtain the mass-size fractal dimension[5] $D$ of CS using the inclined plane. The conservation of energy for a ball of mass $M$ rolling from rest down a ramp (Fig. 1) is expressed as

$$MgH = Mgh + I\omega^2/2 + Mv^2/2, \qquad (1)$$

where $H$ is the initial height, and $h$, $\omega$, and $v$ are, respectively, the height, the angular velocity, and the velocity of translation of the ball at time $t$. In Eq. (1), $g$ is the gravitational field, and $I$ is the moment of inertia about any diameter of the ball. The last quantity is given by[6]

$$I = \frac{2}{3}\int_0^R r^2\, dm, \qquad (2)$$

where $R$ is the radius of the ball. The mass element, $dm$, for a $D$-dimensional ball satisfying the mass($M$)-size($r$) scaling relation $M(r) = kr^D$ is $dm = M(r+dr) - M(r) = k(r+dr)^D - kr^D = kr^D\ (1 + Ddr/r) - kr^D = Dkr^{D-1}\, dr$. After the substitution of the last expression in Eq. (2) and integration we get

$$I = 2DkR^{D+2}/3(D+2). \qquad (3)$$

For a compact sphere of mass $M$ in the physical space we have $D=3$, $k=M/R^3$ and then Eq. (3) reads $I=2MR^2/5$, as expected. Using Eq. (3) in Eq. (1), and assuming that $v=\omega R$ (i.e., the ball rolls down the ramp without slipping) we obtain after a little rearrangement of the terms:

$$S \equiv M\left(\frac{g(H-h)}{v^2} - \frac{1}{2}\right) = \frac{Dk}{3(D+2)} R^D. \qquad (4)$$

Thus, according to the last expression a plot of log $S$ versus log $R$ is a straight line whose slope is the dimension $D$.

The CS in our experiments had a spherical shape and were obtained from random and irreversible compaction of

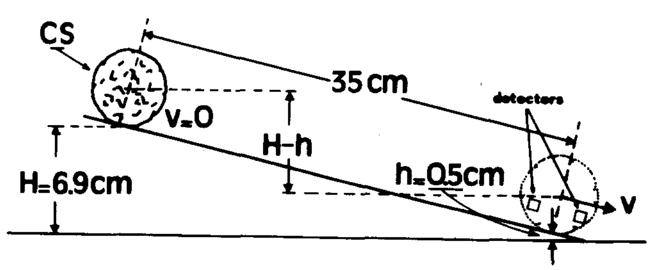


Fig. 1. The inclined plane used in the experiments with the crumpled surfaces (CS). The initial velocity of the CS ball is zero. The velocity $v$ of the CS at the end of the ramp (at the height $h$ from the horizontal) is measured with the aid of two detectors connected with a digital clock (precision of $1\times10^{-3}$ s). See text, second and third paragraphs.