Why Bach Sounds Funny on the Piano

by Guy Vandegrift

String players have many reasons for appreciating the distinction between tempered and natural scales. One of the beauties of stringed instrument is that while they are difficult to play "in tune" with the piano, they can be *more* in tune than a piano. A fine quartet plays harmonies unimaginable from a piano or symphony orchestra.

My viola is occasionally used as a prop in lectures on the physics of music to demonstrate why the natural scale, so pleasing to the ear, needs to be tempered slightly. This demonstration can also be used as an exercise to boost the confidence of any string student, frustrated by the unending struggle for perfect intonation.

The mystery arose soon after my high school viola teacher started me on Bach's First Suite for unaccompanied cello. It begins:

These notes have always sounded strange on a piano. Was it my inexperience with the piano? Does this particular melody demand a stringed instrument for mystical reasons only Bach could understand? Or perhaps, that initial exposure on the viola later spoils the melody for any other instrument. If I had heard the suite first on the piano, would it have then sounded strange on the viola? This puzzle was almost forgotten as it faded into the background of the many questions a high school student can ask.

Fourteen years later, the winter of 1984 was typical for Siberia, but wonderfully alien to a Californian. Only about fifty American and Soviet scientists were allowed to participate in long-term exchanges that year; the Cold War was still raging. My nine-month visit was productive both professionally and personally: life-long friends were made (my wife among them), and eventually two papers on theoretical plasma physics were published. I had plenty of time to play and think about music, and the viola was a big hit with the Soviet scientists.

One typically cold day, a Siberian physicist visited my office to ask if the musical scale could be explained mathematically. I dropped my work for the afternoon, and proceeded to discover a simple exercise for the violin, viola, or cello. The old mystery of Bach's Suite for cello, played on the piano, was finally solved.

To put it bluntly, the piano's tempered scale can never be perfectly in tune. Furthermore, the tempered B-natural is sharper than what the ear expects to hear. Since Bach never said anything about playing a well-tempered cello, I continue to make the B-natural a little bit flat, but now do so with confidence.

The Mathematics of Natural and Tempered Scales

Mathematics can describe the difference between natural and tempered scales, which are so nearly identical that an unsophisticated ear cannot distinguish them. Concert A has a pitch (also called frequency) of 440, meaning that the string vibrates 440 times every second. As any musician knows, some intervals sound consonant, while others are dissonant. Those notes that sound consonant with 440 A do so because they have pitch equal to 440 times what mathematicians call a rational fraction. Such a fraction is of the form p/q, where p and q are integers (e.g., whole numbers such as 0,1,2,3,...).

For example, if p = 2 and q = 3, then p/q = 2/3, and we have the pitch 440 x (2/3) = 293.333, which a string player would recognize as the open D, quite consonant with 440 A. To obtain consonance it is usually necessary that both p and q be small. Thus a rational fraction such as 3/11 would probably not produce consonance because q = 11 is somewhat large. Scales based on pitches defined by these rational fractions (p/q) are called natural scales, an appropriate name because the ear naturally hears these intervals as consonant.

The recognition that we can choose pitch transforms intonation problems to an expression of musical style.

Rational fractions (p/q) also occur in a different musical context. If one person claps two beats in the time it takes another to clap three beats, the two people form the same three-two resonance that occurs when the pitches A and D form a perfect fifth. In other words, a violin's A string vibrates three times in the time it takes the D string to vibrate twice.

The pitches of the tempered scale are obtained from a completely different formula: 440 is multiplied by $2^{n/12}$, where n is a positive or negative integer. To obtain the tempered pitch for open D, set n equal to -7. This yields: $440 \times 2^{-7/12}$ = 293.665—which is slightly higher then the pitch of the natural open D. The two pitches (293.333 and 293.665) differ by only 0.1 percent and are so close that we produce much larger changes in pitch with our vibrato.

If the human ear perceives the natural ratios as being consonant, why would anybody invent the slightly dissonant tempered scale? There are two closely related answers to this question. First, only the tempered scale allows a composer to modulate, or transpose into another key, without distorting the melody. A second answer is that it is impossible to construct a natural scale whereby all the so-called consonant intervals are in tune. A completely natural scale is inherently inconsistent with itself because it demands slightly different pitches for each note, depending on the context in which the note is played.

Math buffs will want to compile a list of all rational fractions such that $1 < p/q < 2$ where $p \leq 6$ and $q \leq 6$ and compare with the list of consonant intervals shown below. They might also count the number of half-tones in an octave and a fifth to see where the numbers seven and twelve came from in the formula for the tempered scale. (The choice of twelve half tones in the tempered scale is somewhat of a mathematical coincidence.)

Those with no love of mathematics can take comfort in the fact that our word algebra is derived from the Arabic word for demon and simply remember that different methods for calculating pitches yield slightly different results. Instead of focusing on the calculation, learn how to use your instrument almost as a calculator to create nearly identical pitches for the "same" note. The following exercise shows how to find two different locations for the first finger E on the open D string. Both locations are appropriate for a natural scale, yet are surprisingly far apart.

The Exercise

You will need your instrument, a friend, and a ruler. In order for violins, violas, and cellos to follow this together, I will focus on the first finger E on the D string, instead of the B-natural in the phrase by Bach. Start by tuning the instrument as best you can. Don't worry about whether you are in tune to the piano or tuning fork. Just make the strings exactly in tune with each other by eliminating the beats.

Consider three notes: the open G, the open A, and the E-natural formed by

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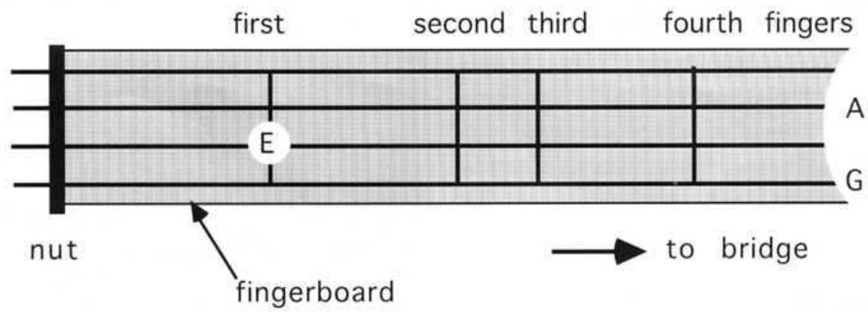
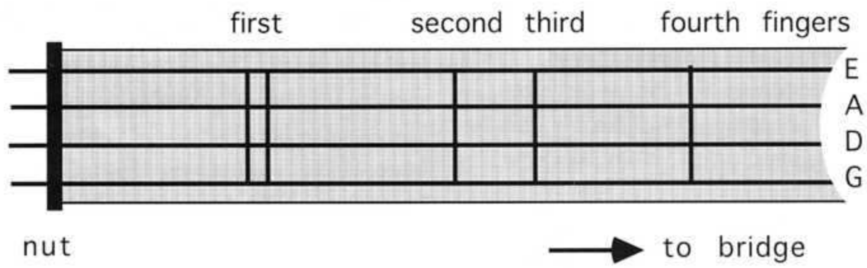
putting the first finger on the D string. We want to find the proper location for the first finger E. These three notes are marked at the right for a violin.

Where does the first finger E belong? Play a double stop between the first finger E and the open G to make a sixth. Tune it exactly. Next play a double stop between the E and the open A to make a fourth. Make this sound exactly in tune. You should notice that the E has to be made slightly flatter when it sounds against the lower string. There are actually two ways to play this E, depending on which string you play it against!

The distance your first finger moves is quite large, although you probably need to watch someone else do it to believe it. A calculation shows that your finger must move a distance equal to the length of the string divided by ninety. For example, the string of a three-quarter-size violin is about 30 centimeters long, so that the finger moves 0.33 centimeters. This is approximately one-eighth of an inch, and the distance is even greater for larger instruments.





To get a visual idea of what this looks like, suppose you are marking finger positions on a beginner's violin (E, F-sharp, G, A on the D string). Only this time you decide to mark two first finger Es. See the fingerboard example above.

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This discovery helped me as a string player in two ways: First, to play a first finger in the tempered scale, check against the higher string because that natural harmony is closer to the tempered scale[1]. Second, I learned that the tempered scale isn't always right for me, and that explains why Bach sounds "funny" on the piano.

The Calculation

To understand how this exercise works, we shall calculate the pitch of the first finger E using two methods. Both are based on natural intervals starting from open G, using ratios (p/q) from the "just" natural scale, shown in the table below.[2] We have already discussed one of these ratios: 3/2 corresponds to going up a fifth.

Minor third	6/5
Major third	5/4
Fourth	4/3
Fifth	3/2
Sixth	5/3
Octave	2/1

The first way to find the pitch of E is to make a sixth with the open G string. From the table, going up a sixth corresponds to a ratio of 5/3. Using an arbitrary pitch of 196 for the open G string, we obtain

$$(196) \times (5/3) = 326.667$$

as one "correct" pitch for E.

Recall from the exercise that the "other" E was obtained by tuning with the open A. To find this pitch from the open G, we go up two fifths to the open A, and then come down a fourth. Going up two fifths introduces two factors of 3/2, and going down a fourth introduces a factor 3/4, which is the inverse of 4/3 (because here we go down a fourth). Thus, the second method yields

$$(196) \times (3/2) \times (3/2) \times (3/4) = 330.750$$

as another "correct" pitch for E.

The differences between the two pitches (approximately 327 and 331) is quite large, about 1 percent. Thus, to obtain an approximate value for difference in placement of the first finger E, one should move about 1 percent of the string length. A more careful calculation shows that the finger must move a distance equal to the string length divided by ninety.

My personal experience suggests that the psychological benefits of this exercise are not insignificant. It proved that I could actually detect those subtle differences in pitch my musically talented friends and teachers claimed to hear effortlessly. I had been playing a natural scale for years without knowing it!

Previously, I viewed intonation as a matter of scientific precision: if one gallon of water is measured to the last drop, then there is usually an error of a few hundredths or even tenths of a drop. Likewise, if a measurement of actual pitch is carried to enough decimal places, a discrepancy with the ideal always emerges. This recognition that we can choose pitch changed that mechanical view, transforming the question of intonation from an unsolvable problem to an expression of musical style and freedom.

REFERENCES

1. This same E can also be played on the G string and tested against the C string on a viola or cello. This yields the same "flattened" E-natural obtained by tuning to the open G string. Similar statements are true for the third-finger-third-position notes on a violin (B on the D string, F-sharp on the A, and C-sharp on the E) when tested against the lower string.
2. The "just" natural scale is described in almost all textbooks on the physics of music. Excellent discussions are found in:

• H. E. White and D. H. White, *Physics and Music* (Philadelphia: Saunders College, 1980), pp. 170–178.
• Johan Sundberg, *The Science of Musical Sounds* (San Diego: Academic Press, 1991), pp. 78–103.
• Jess J. Josephs, *The Physics of Musical Sounds* (Princeton: Van Nostrad, 1967), pp. 78–87.

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