Transverse bending waves and the breaking broomstick demonstration

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When a broomstick is supported at both ends by two wine glasses, a strong downward blow to the center will break the stick, leaving the wine glasses undisturbed, provided care is taken to cushion the wine glasses against an initial and brief downward motion of the ends of the broomstick. This downward motion is analyzed and estimated to be about 1 mm in magnitude. Qualitative experimental evidence of this motion is easily obtained using a force probe to monitor a light and nondestructive tap to a 2-m measuring stick. The method of analysis developed here leads to a simple derivation of the dispersion relation for transverse bending waves on a long rod. © *1997 American Association of Physics Teachers.*

I. INTRODUCTION

Images of transverse bending waves rushed through my mind as I first saw the ''Breaking Broomstick Demonstration'' and wondered if it might all end with broken glass and spilt wine.[1] In this very old demonstration, a broomstick is supported at both ends by pins which rest on two wine glasses, as shown in Fig. 1. A strong downward blow to the center breaks the broomstick, leaving the wine undisturbed as the two broken broomstick halves fold between the wine glasses and fall to the floor. Transverse bending waves[2-6] do
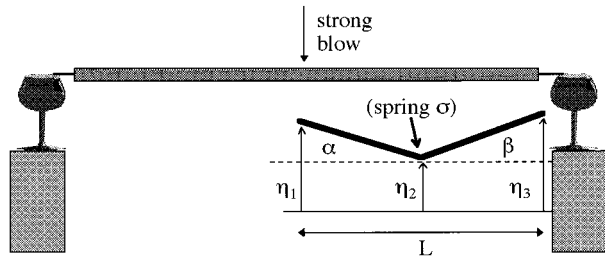
Fig. 1. Sketch and definition of variables for the ''Breaking Broomstick Demonstration.'' The two rigid rods of length $L/2$ are assumed to be connected by a bending spring with spring constant σ.

Fig. 3. Force measured by a probe at end of a ''2-m'' stick. The rod was struck at approximately $t=0$. Two time scales are shown for the same event and measurement.

exert a temporary downward force on the wine glasses, as suggested by Mamola and Pollard,[1] who pointed out the need for support pins to cushion against these waves. The purpose of this paper is to analyze this downward motion in detail by modeling each half of the broken broomstick as two rigid rods connected by a spring mechanism. Also, the method of analysis presented here is extended to produce an easily understood derivation of the dispersion relation for transverse bending waves on a long rod. This derivation illustrates that transverse bending waves and string waves differ in the way potential energy is stored.

To show students why the demonstration *might* work, place a long and narrow block on a table so that one end can be struck sharply in a direction parallel to the surface of the table and perpendicular to the long axis of the block. Place two small objects of different color on opposite sides of the block at the unstruck end. Ask students to predict which object will move as the block is struck. (This needs to be practiced beforehand; if you are not careful, both objects will move.) The applied impulse gives rise to two motions, one linear and the other rotational. It has been shown[1] that the
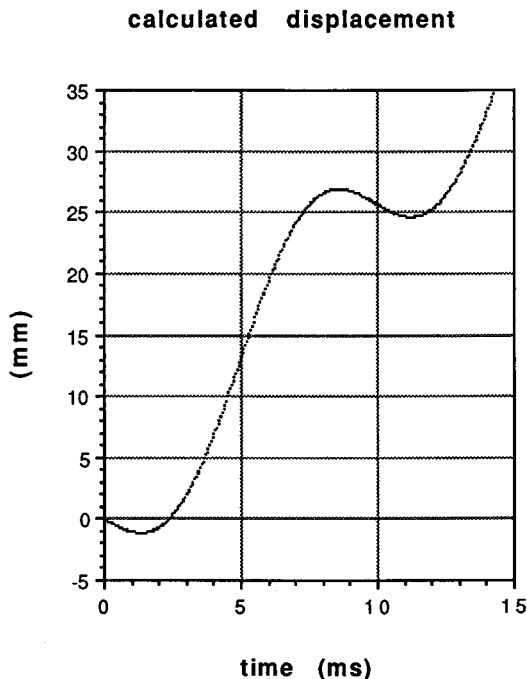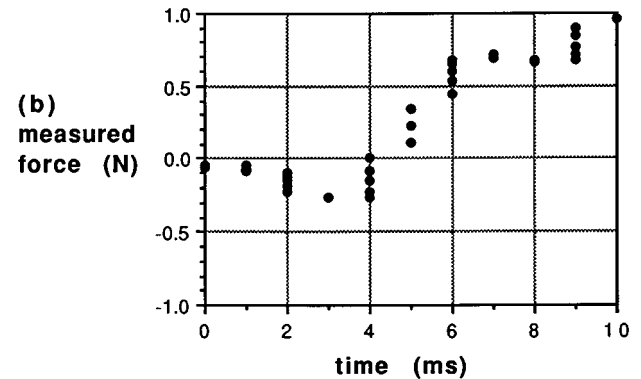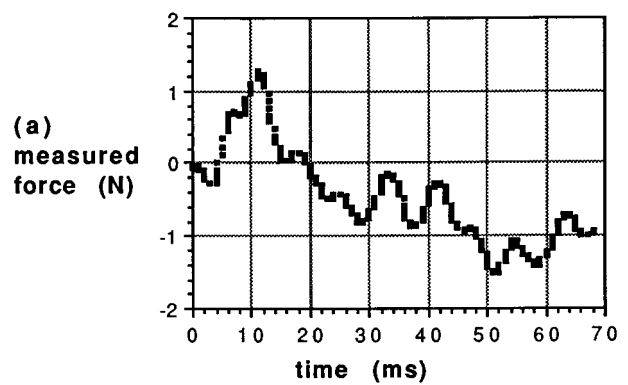
rotational motion dominates at the unstruck end, causing it to go in a direction opposite to the impulse. Therefore the ''Breaking Broomstick Demonstration'' *should* work, assuming the broken broomstick halves remain perfectly rigid as they turn upward and fall to the floor. But no object is perfectly rigid!

Although it is probably best to do this demonstration with a stick that only looks like a broomstick, we shall refer to it as a ''broomstick.'' The problem becomes tractable only if one either assumes that the broomstick is not broken by the rod,[1] or that the broomstick is broken immediately. Here, the latter simplification is made, so that the free motion of a half-broomstick is calculated. This half-broomstick is modeled as two rigid rods connected by a bending-spring mechanism. The calculated result is shown in Fig. 2, where it can be seen that an initial and brief downward displacement of about 1 mm into the wine glass is predicted before the stick moves upwards and away from the wine glass.

Qualitative experimental evidence for the initial downward motion is easily obtained using a force probe available in many university classrooms.[7] The force probe is attached to one end of a 2-m measuring stick which was tapped lightly at the center. The result is shown in Fig. 3, where the similarity between the calculated displacement of Fig. 2 and the measured force in Fig. 3(b) is apparent. These results are fairly consistent with careful experimental observations and calculations made over 40 years ago on steel rods, where the equation of motion for transverse bending waves was solved for initial conditions appropriate for this problem.[3,4]

calculated displacement

Fig. 2. Calculated displacement of the unstruck end of a broomstick.

II. CALCULATION OF INITIAL DOWNWARD DISPLACEMENT

In order to model transverse vibrations, we make some simplifying assumptions about one of the pieces shortly after the stick has been broken. First, we treat the half-stick as an isolated object of length L, subject to a vertical force F at the end where the blow was struck. The opposite end is at the wine glass and taken to be completely free. Bending motion is modeled as that of two perfectly rigid rods connected by a spring mechanism that tends to keep the rods parallel. We shall assume small deviations from the initial position.

Let the two rigid rods have length $L/2$ and mass $M/2$, and let the bending-spring mechanism obey a generalization of Hooke's law, with restoring torque proportional to the angle of bending. Define the transverse coordinates for three points on the half-broomstick by η_1, η_2, η_3, as shown in Fig. 1. We let $F>0$ represent an upward force, so that the applied force can be modeled as a negative impulse. Since the displacements are assumed small, the angles are approximated as

$$\alpha = \frac{\eta_1 - \eta_2}{L/2}, \quad \beta = \frac{\eta_3 - \eta_2}{L/2}. \tag{1}$$

With three variables (η_1, η_2, η_3), we need three equations of motion:

$$F = M\left(\frac{\ddot{\eta}_1 + 2\ddot{\eta}_2 + \ddot{\eta}_3}{4}\right), \tag{2a}$$

$$\left(\frac{L}{2}\right)F = \frac{1}{3}\left(\frac{M}{2}\right)\left(\frac{L}{2}\right)^2[\ddot{\alpha} - \ddot{\beta}], \tag{2b}$$

$$\frac{1}{3}\left(\frac{M}{2}\right)\left(\frac{L}{2}\right)^2\ddot{\beta} = -\sigma(\alpha + \beta) - \frac{1}{2}\frac{L}{2}\frac{M}{2}\ddot{\eta}_2. \tag{2c}$$

The first equation (2a) is Newton's second law for an object of mass M, whose center-of-mass coordinates are $(\eta_1 + 2\eta_2 + \eta_3)/4$. The second equation (2b) is a torque equation about the center of mass of the entire system. (The spring mechanism does not enter here because only external torques contribute.) The third equation (2c) describes rotational motion of the *unstruck* rod about the pivot point connecting the two rods. It contains two torquelike terms on the right-hand side (RHS). The first term, $-\sigma(\alpha + \beta)$, is the torque caused by the spring mechanism where σ is the generalized spring constant. The second term on the RHS of (2c) is an inertial "pseudo-torque" proportion to $d^2\eta_2/dt^2$. To understand this term, note that a pseudo-force of magnitude $ma = (M/2)a$ acts at the center of mass of the unstruck rod, where a is the acceleration of the pivot point. This "pseudo-torque" can be used to shut a car door by carefully backing up, and then slamming on the brakes. It is analogous to the "centrifugal force" and can be derived from Newton's second law as seen in an accelerating reference frame.[8]

While manipulation of these equations is tedious, they are linear, and can be simplified by a transformation of variables from (η_1, η_2, η_3) to (B, R, T), where

$$B = \eta_1 - 2\eta_2 + \eta_3,$$
$$R = \eta_1 - \eta_3, \tag{3}$$
$$T = \eta_1 + 2\eta_2 + \eta_3$$

are new variables designed to represent bending, rotation, and translation of the center of mass, respectively. The linear transformation back from (B, R, T) to (η_1, η_2, η_3) is easily obtained by finding the inverse of a 3×3 matrix, or by using algebra.

Using the new (B, R, T) variables, the three equations of motion are quite simple:

$$M\ddot{B} + \frac{192\sigma}{L^2}B = 12F, \tag{4a}$$

$$M\ddot{R} = 6F, \tag{4b}$$

$$M\ddot{T} = 4F. \tag{4c}$$

What makes the variables (B, R, T) special is that they represent the three normal modes of the system. They also facilitate a derivation of (4) that does not require the "pseudo-torque." From (3) and also (7) and (10) below, the Lagrangian can be constructed from the potential energy $U = 2\sigma B^2/L^2 - \int Fd\,\eta_3$, where η_3 as well as kinetic energy K must be expressed in terms of the (B, R, T) variables. For example, the kinetic energy takes on a simple form, $K = M[3\dot{T}^2 + 4\dot{R}^2 + \dot{B}^2]/96$, where it is worth noting that cross terms such as $\dot{B}\dot{R}$ are mercifully absent.

Regardless of whether we use a "pseudo-torque" or Lagrangian formalism, Eq. (4) is easily solved if we let the downward force be an impulse, $F \propto -\delta(t)$, so that R (rotation) and T (translation) describe free particles, while the variable B (bending) acts like a simple harmonic oscillator. The impulse gives each variable (B, R, T) initial conditions. After finding (B, R, T) for such an impulse, the variables are converted back to (η_1, η_2, η_3). The result for the end near the wine glass $\eta_3(t)$ is

$$\eta_3 = V_A t - \frac{3V_A}{2\omega}\sin(\omega t), \tag{5}$$

where $\omega = (192\sigma)^{1/2}L^{-1}$ is the natural frequency of bending motion and the magnitude of the impulse force is defined so that V_A is the time-averaged velocity of the unstruck end. Thus $\eta_3 = V_A t$ represents the motion up and away from the wine glass which makes the demonstration work. The small-angle approximations preclude modeling of rod rotation beyond this stage. The second term in (5) oscillates in time and is associated with bending motion.

Equation (5) is used to plot $\eta_3(t)$ for realistic values of ω and V_A in Fig. 2. The value of $V_A = 2.6$ m/s was obtained from Ref. 1, which contains a picture made from a videotape of the demonstration. From Fig. 3 of that reference, one sees that the stick rises approximately 4.3 cm during the first 1/60 of a second. To obtain ω, we use the oscillation frequency of the lowest-order mode ($n=1$) of a free vibrating rod:[2]

$$\omega = \nu_s bk^2 \cong \nu_s b\left(\frac{n+1/2}{L}\pi\right)^2, \tag{6}$$

where the radius of gyration, $b = R/2 \approx 6.3$ mm, for a cylinder of radius R. The length L of a typical half-broomstick is taken to be 0.825 m.

Equation (6) is only an approximation. As seen from the dispersion relation (11) below, there are four values of wave number k for a given frequency ω: two real and two imaginary. The actual lowest-order wave function for a free rod is a superposition of $\cos(kx)$ and $\cosh(kx) = \cos(ikx)$, with k being selected to satisfy a boundary condition somewhat more complicated than that associated with common string waves.[2]

The speed of sound in wood is typically between 1300 and 4700 m/s, so we take an average value of $\nu_s = 3000$ m/s for

our calculation.[9] This yields $\omega \approx 640$ s^{-1}. From either Fig. 2 or Eq. (5), the maximum downward displacement of the broomstick is $0.28V_A/\omega \approx 1$ mm, with perhaps a factor of 2 uncertainty due to lack of knowledge of the sound speed in wood, and due to the fact that higher order modes are excluded by the model.

We are now in a position to check our assumption that the broomstick was immediately broken by the blow. First, we estimate the speed of the rod which strikes the blow. A baseball thrown at a speed of 14 m/s will travel 20 m, a distance easily achieved with only an arm and wrist motion. Such a hand speed would be approximately doubled by the lever arm of a rod. I believe the rod speed is larger because the chopping motion is very comfortable for the arms and back, and because the wrist is able to rotate the rod effectively. Since the person doing the demonstration is in no mood to strike a light blow, it is reasonable to assume that the broken ends of the broom at the center moved down more than 2.8 cm during the first millisecond.

It is interesting to compare other measurements with my estimate of rod speed (28 m/s), which is twice the speed of a baseball thrown without a wrist snap. Investigations of karate blows indicate hand speeds of 5–10 m/s.[10–13] A lower bound on the rod speed can be obtained from the observed 20-m/s speed of a sledgehammer used to break the block in the ''Bed of Nails Demonstration.''[14]

From Fig. 2, we see that the broomstick might be in the process of being broken when the downward motion occurs at the wine glasses. This confusion as to when the broomstick breaks adds further uncertainty to the calculated value of the downward displacement.

III. OBSERVATIONS WITH FORCE PROBE AND MEASURING STICK

A related demonstration is simple to set up if one has a force probe compatible with motion detectors often used in the teaching of introductory physics.[7] It shows students that there is a brief downward force on the broomstick. And it shows that there will be a large and not very brief downward motion onto the wine glasses in the event that the blow does not break the broomstick.

The broomstick is replaced by a wooden ''2-m'' measuring stick made of hard wood (cross section 8×25 mm). A light tap to the center of the measuring stick will excite a spectrum of bending modes. One end of the measuring stick is attached to the force probe which collects data continuously and is self-triggered in such a way as to record data prior to experiencing any force above a threshold value selectable by the user of the software.

From Fig. 3(a), we see that the measuring stick is again pushing down after 20 ms. This can be attributed to the stick's lowest-order mode, which undergoes approximately a quarter-cycle in Fig. 3(a). The observed 240-ms period is consistent with a sound speed of 4600 m/s in the wood, provided we take the wavelength $2\pi/k$ to be 4 m. This lowest-order mode is absent in both the mathematical model and the actual demonstration because the broomstick has been broken into two pieces before this low-frequency mode undergoes one cycle.

It is also worthwhile to let students shake the measuring stick and feel how low the fundamental frequency is. A similar exercise is to hold a 12-in. ruler down near the edge of a table and see how the free end vibrates as one makes it

shorter. Frequency depends very strongly on wavelength, as can be seen from the dispersion relation: $\omega \propto k^2 \propto$(wavelength)$^{-2}$.

IV. DERIVATION OF WAVE EQUATION

The model used here can be extended to derive the wave equation for transverse bending modes. My derivation is similar to that of Crawford[5] in that he treats the system as a large number of discrete and rigid elements connected by massless springs. However, Crawford's derivation requires an extra set of springs and motions not needed here. In contrast, the derivations found in most textbooks are difficult to grasp because they involve equations of motion for differential elements.[2]

We consider a long chain of N rigid rods of length $l = L/N$ and mass $m = M/N$, each connected by a mechanism with generalized spring constant σ. Neglecting a small error at the first and last rods, the kinetic energy K is due to motion of the center of mass, plus rotational motion:

$$K = \frac{1}{2} m \sum_{j=1}^{N} \left(\frac{\dot{\eta}_j + \dot{\eta}_{j+1}}{2} \right)^2 + \sum_{j=1}^{N} \frac{1}{2} \frac{ml^2}{12} \left(\frac{\dot{\eta}_j - \dot{\eta}_{j+1}}{l} \right)^2. \tag{7}$$

Since $|\eta_j| \approx |\eta_{j+1}| \ll |\eta_j - \eta_{j+1}|$, we can drop the rotation term and approximate the first term as

$$K \approx \frac{1}{2} m \sum \left(\frac{\dot{\eta}_j + \dot{\eta}_{j+1}}{2} \right)^2 \approx \frac{m}{2} \sum (\dot{\eta}_j)^2$$

$$\rightarrow \frac{\lambda}{2} \int [\dot{\eta}(s)]^2 ds, \tag{8}$$

where the last term is taken in the limit that $N \rightarrow \infty$, and λ is the mass density per unit length. The potential energy is found by observing that the difference between two adjacent rods is

$$\Delta \alpha = \frac{\eta_{j-1} - \eta_j}{l} - \frac{\eta_j - \eta_{j+1}}{l} = \frac{\eta_{j-1} - 2\eta_j + \eta_{j+1}}{l}, \tag{9}$$

so that

$$U = \sum \frac{1}{2} \sigma (\Delta \alpha)^2 = \frac{\sigma}{2} \sum \left(\frac{\eta_{j-1} - 2\eta_j + \eta_{j+1}}{l} \right)^2$$

$$\rightarrow \frac{\sigma l}{2} \int \left[\frac{d^2 \eta(s)}{ds^2} \right]^2 ds. \tag{10}$$

At this point, there are three paths to a dispersion relation or wave equation. The ''quick and dirty'' derivation does not require knowledge of Lagrangian mechanics, but yields only a dispersion relation. One simply notes that, for standing waves, $d^2\eta/dt^2 = -\omega^2 \eta$ and $d^2\eta/ds^2 = -k^2 \eta$. Also, a standing wave's maximum potential and kinetic energy are equal, $U = K$, where U is evaluated at maximum η (when $d\eta/dt = 0$) and K is evaluated at maximum $d\eta/dt$ (when $\eta = 0$). With the replacement of differential operators by ω or k, the dispersion relation follows from equating the integral forms (8) and (10) for T and U, respectively, and noting that everything factors out except

$$\omega^2 = \frac{\sigma l}{\lambda} k^4, \tag{11}$$
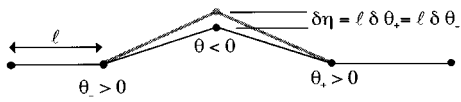
from which the wave equation,

Fig. 4. An upward virtual displacement of the central bending mechanism gives rise to a downward force. The bending angles going from left to right are $(\theta_-, \theta, \theta_+)$ and are defined so that $\theta=0$ for no bending, and θ is negative, while both θ_- and θ_+ are positive. The angles θ_- and θ_+ change by an amount $\delta\theta=\delta\eta/l$, and θ changes by twice that amount.

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\sigma l}{\lambda} \frac{\partial^4 \eta}{\partial s^4} \tag{12}$$

can be guessed or postulated, but not, in my view, rigorously derived.

In the unlikely event that you find yourself doing this demonstration to a physics class studying the last chapter of Goldstein,[15] the most elegant way to derive the wave equation is to note that the Lagrangian is expressed as an integral of the Lagrangian density: $L=\int \mathscr{L}(\partial\eta/\partial t, \partial^2\eta/\partial s^2)ds$. A slightly more pedestrian but equally rigorous derivation is to obtain finite difference equations from the Lagrangian, $L=K-U$, as expressed by the *sums* in (8) and (10). The algebra is straightforward but tedious. Two key steps will illustrate the method.[15]

$$\frac{\partial U}{\partial \eta_n} = \frac{\partial}{\partial \eta_n} \left\{ \frac{\sigma}{2} \sum_{j=1}^{N} \left(\frac{\eta_{j-1} - 2\eta_j + \eta_{j+1}}{l} \right)^2 \right\}$$

$$= \frac{\partial}{\partial \eta_n} \{\text{three terms in the sum}\}$$

$$= \frac{\sigma}{2} \frac{\partial}{\partial \eta_n} \left\{ \left(\frac{\eta_{n-2} - 2\eta_{n-1} + \eta_n}{l} \right)^2 \right.$$

$$+ \left(\frac{\eta_{n-1} - 2\eta_n + \eta_{n+1}}{l} \right)^2$$

$$\left. + \left(\frac{\eta_n - 2\eta_{n-1} + \eta_{n-2}}{l} \right)^2 \right\} \tag{13a}$$

and

$$\lim_{l \to 0} \left(\frac{\eta_{n+2} - 4\eta_{n+2} + 6\eta_{n+2} - 4\eta_{n+2} + \eta_{n+2}}{l^4} \right) = \frac{\partial^4 \eta}{\partial s^4}. \tag{13b}$$

There is also a derivation of the wave equation that does not use Lagrangian mechanics. While accessible to the widest audience, this derivation is actually the most difficult. The force F on a bending-spring mechanism can be calculated by making a virtual displacement $\delta\eta$ of one mechanism, keeping all other mechanisms motionless (but allowing them to bend). The energy of bending associated with this displacement is $F\delta\eta = \Sigma \sigma \delta\theta_j$, where the sum, Σ, is over the three mechanisms that bend when one mechanism moves. From Fig. 4, one obtains a downward restoring force of $(\theta_- - 2\theta + \theta_+)(\sigma/1)$, where $\theta_\mp$ is the bending angle of the mechanism to the (left/right), and θ is the bending angle of the mechanism that moves. Since $(\theta_- - 2\theta + \theta_+) \propto \partial^2\theta/\partial s^2$ and $\theta \propto \partial^2\eta/\partial s^2$, we see that $F \propto \partial^2\eta/\partial t^2 \propto -\partial^4\theta/\partial s^4$. The wave equation follows by assuming that all the mass is located in the bending mechanisms. What makes this derivation ''uncomfortable'' is the fact that a displacement of one and only
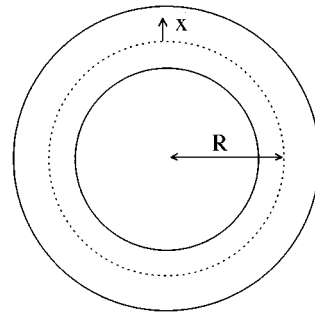

Fig. 5. Schematic drawing of a very long rod bent into a circle of radius R. The dotted circle represents matter neither compressed nor extended by the bending. Matter outside the dotted circle is extended while matter inside the dotted circle is compressed. The distance from the dotted line is x.

one spring mechanism is impossible for finite $\delta\eta$. The simplest resolution of this problem is to allow the spring mechanism added motion that allows for slight elongation of the rods.

Our final task is to relate the idealized system of rigid rods and massless springs to a physical rod consisting of material with Young's modulus E, and mass density ρ. Suppose a very long rod is bent into a circle. Using the model of rigid rods, we have N spring mechanisms, each bent by an angle of $2\pi/N$, so that the potential energy is

$$U = N \frac{\sigma}{2} \left(\frac{2\pi}{N} \right)^2. \tag{14}$$

Next, we bend a physical rod into a large circle of radius $R=L/2\pi$. Each atom of the rod is compressed or expanded by an amount ΔL, where $\Delta L/L = x/R$ with x being the distance to the dotted line shown Fig. 5. The force of compression at an element of surface da is $E(\Delta L/L)da$, and the energy required to attain this compression is $(E/2L)(\Delta L)^2 da$. Integrating over a cross section of the rod, we have

$$U = \frac{E}{2} \int \frac{(\Delta L)^2}{L} da = \frac{E}{2} \left(\frac{4\pi^2}{L} \right) \int x^2 \, da. \tag{15}$$

The mass density per unit length λ, and mass density per unit volume are related by $\lambda = \rho \int da$. Using the equality of the two forms of potential energy (14) and (15), one obtains

$$\frac{\sigma l}{\lambda} = \frac{E}{\rho} \frac{\int x^2 \, da}{\int da} \equiv \frac{E}{\rho} b^2, \tag{16}$$

which defines the radius of gyration, b. Insertion of (16) into (12) yields the usual wave equation for transverse bending modes.

The location of the dotted line in Fig. 5 and hence the origin of x has not been defined. This problem is easily remedied by minimizing the potential energy required to bend the long rod into a circle. We must choose x_0, the location of the dotted line, so as to minimize the integral $\int (x - x_0)^2 da$. Setting to zero the derivative of this integral with respect to x_0, we see that $\int x da = \int x_0 da$, so that $\int x da = 0$ for a proper choice of origin ($x_0 = 0$).

V. TRANSVERSE STRING WAVES

It is instructive to compare transverse bending waves in a rod with transverse tension waves on a string. The formula

(8) for kinetic energy K is the same for both systems, but the potential energy (10) U is different: $U = \tau \Delta L$, where τ is the tension and ΔL is the change in length of the string,

$$L + \Delta L = \int d\ell = \int \sqrt{ds^2 + d\eta^2} \approx \int \left[1 + \frac{1}{2} \left(\frac{d\eta}{ds} \right)^2 \right] ds, \tag{17}$$

where we have Taylor expanded the small term in the radical since $|ds| \ll |d\eta|$ for low amplitude waves. Thus, for a string wave's potential energy,

$$U = \frac{\tau}{2} \int \left(\frac{\partial \eta}{\partial s} \right)^2 ds. \tag{18}$$

One might say that potential energy in the transverse wave of a string is stored as a change in string length, while energy in a transverse bending wave is stored in the bending, or curvature of the rod.

Comparison of (10) and (18) shows that the dependence of potential energy on wave number is much stronger for bending waves ($U \propto k^4$) than for transverse string waves ($U \propto k^2$). Students who have carried long boards or fishing rods may have noticed that long objects seem far from rigid; long sticks feel ''wobbly.'' This lack of rigidity is closely related to the low frequency of the fundamental mode. Both arise from the low potential energy associated with the bending of very long objects.

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